GLOBAL REGULARITY FOR SOME CLASSES OF LARGE SOLUTIONS TO THE NAVIER-STOKES EQUATIONS

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Abstract. In [4]-[6] classes of initial data to the three dimensional, incompressible Navier-Stokes equations were presented, generating a global smooth solution although the norm of the initial data may be chosen arbitrarily large. The main feature of the initial data considered in [6] is that it varies slowly in one direction, though in some sense it is "well prepared" (its norm is large but does not depend on the slow parameter). The aim of this article is to generalize the setting of [6] to an "ill prepared" situation (the norm blows up as the small parameter goes to zero). As in [4]-[6], the proof uses the special structure of the nonlinear term of the equation.

1. Introduction

We study in this paper the Navier-Stokes equation with initial data which are slowly varying in the vertical variable. More precisely we consider the system

$$\begin{cases} \partial_t u + u\cdot\nabla u - \Delta u = -\nabla p \quad \text{in} \quad \mathbb{R}^+ \times \Omega \\ \operatorname{div} u = 0 \\ u|_{t=0} = u_{0,\varepsilon}, \end{cases}$$

where $\Omega = \mathbb{T}^2 \times \mathbb{R}$ (the choice of this particular domain will be explained later on) and $u_{0,\varepsilon}$ is a divergence free vector field, whose dependence on the vertical variable x_3 will be chosen to be "slow", meaning that it depends on εx_3 where ε is a small parameter. Our goal is to prove a global existence in time result for the solution generated by this type of initial data, with no smallness assumption on its norm.

1.1. **Recollection of some known results on the Navier-Stokes equations.** The mathematical study of the Navier-Stokes equations has a long history, which we shall describe briefly in this paragraph. We shall first recall results concerning the main global wellposedness results, and some blow-up criteria. Then we shall concentrate on the case when the special algebraic structure of the system is used, in order to improve those previous results.

To simplify we shall place ourselves in the whole euclidian space $\mathbb{R}^d$ or in the torus $\mathbb{T}^d$ (or in variants of those spaces, such as $\mathbb{T}^2\times\mathbb{R}$ in three space dimensions); of course results exist in the case when the equations are posed in domains of the euclidian space, with Dirichlet boundary conditions, but we choose to simplify the presentation by not mentioning explicitly those studies (although some of the theorems recalled below also hold in the case of domains up to obvious modifications of the statements and sometimes much more difficult proofs).

Key words and phrases. Navier-Stokes equations, global wellposedness.

1.1.1. *Global wellposedness and blow-up results.* The first important result on the Navier-Stokes system was obtained by J. Leray in his seminal paper [21] in 1933. He proved that any finite energy initial data (meaning square-integrable data) generates a (possibly non unique) global in time weak solution which satisfies an energy estimate; he moreover proved in [22] the uniqueness of the solution in two space dimensions. Those results use the structure of the nonlinear terms, in order to obtain the energy inequality. He also proved the uniqueness of weak solutions in three space dimensions, under the additional condition that one of the weak solutions has more regularity properties (say belongs to $L^2(\mathbb{R}^+;L^\infty)$: this would now be qualified as a "weak-strong uniqueness result"). The question of the global wellposedness of the Navier-Stokes equations was then raised, and has been open ever since. We shall now present a few of the historical landmarks in that study.

The Fujita-Kato theorem [10] gives a partial answer to the construction of a global unique solution. Indeed, that theorem provides a unique, local in time solution in the homogeneous Sobolev space $\dot H^{\frac d2-1}$ in d space dimensions, and that solution is proved to be global if the initial data is small in $\dot H^{\frac d2-1}$ (compared to the viscosity, which is chosen equal to one here to simplify). The result was improved to the Lebesgue space L^d by F. Weissler in [32] (see also [13] and [16]). The method consists in applying a Banach fixed point theorem to the integral formulation of the equation, and was generalized by M. Cannone, Y. Meyer and F. Planchon in [1] to Besov spaces of negative index of regularity. More precisely they proved that if the initial data is small in the Besov space $\dot B^{-1+\frac dp}_{p,\infty}$ (for $p<\infty$), then there is a unique, global in time solution. Let us emphasize that this result allows to construct global solutions for strongly oscillating initial data which may have a large norm in $\dot H^{\frac d2-1}$ or in L^d. A typical example in three space dimensions is

$$u_0^\varepsilon(x)\stackrel{\rm def}{=}\varepsilon^{-\alpha}\sin\Big(\frac{x_3}{\varepsilon}\Big)\big(-\partial_2\varphi(x),\partial_1\varphi(x),0\big),$$

where $0<\alpha<1$ and $\varphi\in\mathcal{S}(\mathbb{R}^3;\mathbb{R})$. This can be checked by using the definition of Besov norms:

$$\forall s>0,\ \forall(p,q)\in[1,\infty],\quad \|f\|_{\dot B^{-s}_{p,q}}\stackrel{\rm def}{=}\Big\|t^{\frac s2}\|e^{t\Delta}f\|_{L^p}\Big\|_{L^q(\mathbb{R}^+;\frac{dt}{t})}.$$

More recently in [17], H. Koch and D. Tataru obtained a unique global in time solution for data small enough in a more general space, consisting of vector fields whose components are derivatives of BMO functions. The norm in that space is given by

$$\|u_0\|^2_{BMO^{-1}}\stackrel{\rm def}{=}\sup_{t>0}t\|e^{t\Delta}u_0\|^2_{L^\infty}+\sup_{\substack{x\in\mathbb{R}^d\\ R>0}}\frac1{R^d}\int_{P(x,R)}|(e^{t\Delta}u_0)(t,y)|^2dy,\tag{1.1}$$

where $P(x,R)$ stands for the parabolic set $[0,R^2]\times B(x,R)$ while $B(x,R)$ is the ball centered at x, of radius R.

One should notice that spaces where global, unique solutions are constructed for small initial data, are necessarily scaling-invariant spaces: thus all those spaces are invariant under the invariant transformation for Navier-Stokes equation $u_\lambda(t,x)=\lambda u(\lambda^2t,\lambda x)$. Moreover it can be proved (as observed for instance in [5]) that if B is a Banach space continuously included in the space $\mathcal{S}'$ of tempered distributions such that

$$\text{for any }(\lambda,a)\in\mathbb{R}^+_\star\times\mathbb{R}^d,\quad \|f(\lambda(\cdot-a))\|_B=\lambda^{-1}\|f\|_B,$$

then $\|\cdot\|_B \le C \sup_{t>0} t^{\frac12} \|e^{t\Delta} u_0\|_{L^\infty}$. One recognizes on the right-hand side of the inequality the $\dot B^{-1}_{\infty,\infty}$ norm, which is slightly smaller than the BMO^{-1} norm recalled above in (1.1): indeed the BMO^{-1} norm takes into account not only the $\dot B^{-1}_{\infty,\infty}$ information, but also the fact that first Picard iterate of the Navier-Stokes equations should be locally square integrable in space and time. It thus seems that the Koch-Tataru theorem is optimal for the wellposedness of the Navier-Stokes equations. This observation also shows that if one wants to go beyond a smallness assumption on the initial data to prove the global existence of unique solutions, one should check that the $\dot B^{-1}_{\infty,\infty}$ norm of the initial data may be chosen large.

To conclude this paragraph, let us remark that the fixed-point methods used to prove local in time wellposedness for arbitrarily large data (such results are available in Banach spaces in which the Schwartz class is dense, typically $\dot B^{-1+\frac{d}{p}}_{p,q}$ for finite p and q) naturally provides blow-up criteria. For instance, one can prove that if the life span of the solution is finite, then the $L^q([0,T];\dot B^{-1+\frac{d}{p}+\frac{2}{q}}_{p,q})$ norm blows up as T approaches the blow up time. A natural question is to ask if the $\dot B^{-1+\frac{d}{p}}_{p,q}$ norm itself blows up. Progress has been made very recently on this question, and uses the specific structure of the equation, which was not the case for the results presented in this paragraph. We therefore postpone the exposition of those results to the next paragraph.

We shall not describe more results on the Cauchy problem for the Navier-Stokes equations, but refer the interested reader for instance to the monographs [20] and [24] for more details.

1.1.2. *Results using the specific algebraic structure of the equation.* If one wishes to improve the theory on the Cauchy problem for the Navier-Stokes equations, it seems crucial to use the specific structure of the nonlinear term in the equations, as well as the divergence-free assumption. Indeed it was proved by S. Montgomery-Smith in [25] (in a one-dimensional setting, which was later generalized to a 2D and 3D situation by two of the authors in [12]) that some models exist for which finite time blow up can be proved for some classes of large data, despite the fact that the same small-data global wellposedness results hold as for the Navier-Stokes system. Furthermore, the generalization to the 3D case in [12] shows that some large initial data which generate a global solution for the Navier-Stokes equations (namely the data of [4] which will be presented below) actually generate a blowing up solution for the toy model.

In this paragraph, we shall present a number of wellposedness theorems (or blow up criteria) which have been obtained in the past and which specifically concern the Navier-Stokes equations. In order to make the presentation shorter, we choose not to present a number of results which have been proved by various authors under some additional geometrical assumptions on the flow, which imply the conservation of quantities beyond the scaling (namely spherical, helicoidal or axisymmetric conditions). We refer for instance to [18], [19], [28], or [31] for such studies.

To start with, let us recall the question asked in the previous paragraph, concerning the blow up of the $\dot B^{-1+\frac{d}{p}}_{p,q}$ norm at blow-up time. A typical example of a solution with a finite $\dot B^{-1+\frac{d}{p}}_{p,q}$ norm at blow-up time is a self-similar solution, and the question of the existence of such solutions was actually addressed by J. Leray in [21]. The answer was given 60 years later by

J. Neças, M. Ruzička and V. Şverák in [26]. By analyzing the profile equation, they proved that there is no self-similar solution in L^3 in three space dimensions. Later L. Escauriaza, G. Seregin and V. Şverák were able to prove more generally that if the solution is bounded in L^3, then it is regular (see [9]): in particular any solution blowing up in finite time must blow up in L^3.

Now let us turn to the existence of large, global unique solutions to the Navier-Stokes system in three space dimensions.

An important example where a unique global in time solution exists for large initial data is the case where the domain is thin in the vertical direction (in three space dimensions): that was proved by G. Raugel and G. Sell in [29] (see also the paper [15] by D. Iftimie, G. Raugel and G. Sell). Another example of large initial data generating a global solution was obtained by A. Mahalov and B. Nicolaenko in [23]: in that case, the initial data is chosen so as to transform the equation into a rotating fluid equation (for which it is known that global solutions exist for a sufficiently strong rotation).

In both those examples, the global wellposedness of the two dimensional equation is an important ingredient in the proof. Two of the authors also used such a property to construct in [4] an example of periodic initial data which is large in $\dot{B}^{-1}_{\infty,\infty}$ but yet generates a global solution. Such an initial data is given by

$$u_0^N(x) \stackrel{\text{def}}{=} (Nu_h(x_h)\cos(Nx_3), -\operatorname{div}_h u_h(x_h)\sin(Nx_3)),$$

where $\|u_h\|_{L^2(\mathbb{T}^2)} \leq C(\ln N)^{\frac{1}{4}}$, and its $\dot{B}^{-1}_{\infty,\infty}$ norm is typically of the same size. This was generalized to the case of the space $\mathbb{R}^3$ in [5].

Similarly in [6], that fact was used to prove a global existence result for large data which are slowly varying in one direction. More precisely, if $(v_0^h, 0)$ and w_0 are two smooth divergence free "profile" vector fields, then they proved that the initial data

$$u_{0,\varepsilon}(x_h, x_3) \stackrel{\text{def}}{=} (v_0^h(x_h, \varepsilon x_3), 0) + (\varepsilon w_0^h(x_h, \varepsilon x_3), w_0^3(x_h, \varepsilon x_3)) \tag{1.2}$$

generates, for ε small enough, a global smooth solution. Here, we have denoted $x_h = (x_1, x_2)$. Using for instance the language of geometrical optics in the context of fast rotating incompressible fluids, and thinking of the problem in terms of convergence to the two dimensional situation, this case can be seen as a "well prepared" case. We shall be coming back to that example in the next paragraph.

As a conclusion of this short (and of course incomplete) survey, let us present some results for the Navier-Stokes system with viscosity vanishing in the vertical direction. Analogous results to the classical Navier-Stokes system in the framework of small data are proved in [3], [14], [27] and [8]). To circumvent the difficulty linked with the absence of vertical viscosity, the key idea, which will be also crucial here (see for instance the proof of the second estimate of Proposition 2.1) is the following: the vertical derivative ∂_3 appears in the nonlinear term of the equation with the prefactor u_3, which has some additional smoothness thanks to the divergence free condition which states that $\partial_3 u_3 = -\partial_1 u_1 - \partial_2 u_2$.

1.2. **Statement of the main result.** In this work, we are interested in generalizing the situation (1.2) to the ill prepared case: we shall investigate the case of initial data of the form

$$\Big(v_0^h(x_h,\varepsilon x_3),\frac{1}{\varepsilon}v_0^3(x_h,\varepsilon x_3)\Big),$$

where x_h belongs to the torus $\mathbb{T}^2$ and x_3 belongs to $\mathbb{R}$. The main theorem of this article is the following.

Theorem 1. *Let a be a positive number. There are two positive numbers ε_0 and η such that for any divergence free vector field v_0 satisfying*

$$\|e^{a|D_3|}v_0\|_{H^4}\leq\eta,$$

then, for any positive ε smaller than ε_0, the initial data

$$u_{0,\varepsilon}(x)\stackrel{\text{def}}{=}\Big(v_0^h(x_h,\varepsilon x_3),\frac{1}{\varepsilon}v_0^3(x_h,\varepsilon x_3)\Big)$$

generates a global smooth solution of (NS) on $\mathbb{T}^2\times\mathbb{R}$.

Remarks

- Such an initial data may be arbitrarily large in $\dot B^{-1}_{\infty,\infty}$, more precisely of size ε^{-1}. Indeed it is proved in [6], Proposition 1.1, that if f and g are two functions in $\mathcal{S}(\mathbb{T}^2)$ and $\mathcal{S}(\mathbb{R})$ respectively, then $h^\varepsilon(x_h,x_3)\stackrel{\text{def}}{=}f(x_h)g(\varepsilon x_3)$ satisfies, if ε is small enough,
$$\|h^\varepsilon\|_{\dot B^{-1}_{\infty,\infty}}\geq\frac{1}{4}\|f\|_{\dot B^{-1}_{\infty,\infty}}\|g\|_{L^\infty}.$$
- As in the well prepared case studied in [6] and recalled in the previous paragraph, the structure of the nonlinear term will have a crucial role to play in the proof of the theorem.
- The reason why the horizontal variable is restricted to a torus is to be able to deal with very low horizontal frequencies: as it will be clear in the proof of the theorem, functions with zero horizontal average are treated differently to the others, and it is important that no small horizontal frequencies appear other than zero. In that situation, we are able to solve globally in time the equation (conveniently rescaled in ε) for small analytic-type initial data. We recall that in that spirit, some local in time results for Euler and Prandtl equation with analytic initial data can be found in [30]. In this paper we shall follow a method close to a method introduced in [2].
- We finally note that we can add to our initial data any small enough data in $\dot H^{\frac{1}{2}}$, and we still obtain the global existence of the solution. Indeed, by the results contained in [11], if we fix an initial data which gives a global in time solution, then, all initial data in a small neighborhood, give global in time solutions.

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2. Structure of the proof

2.1. **Reduction to a rescaled problem.** We look for the solution under the form

$$u_\varepsilon(t,x) \stackrel{\text{def}}{=} \Big(v^h(t,x_h,\varepsilon x_3), \frac{1}{\varepsilon}v^3(t,x_h,\varepsilon x_3)\Big).$$

This leads to the following rescaled Navier-Stokes system.

$$(RNS_\varepsilon)\quad \begin{cases} \partial_t v^h - \Delta_\varepsilon v^h + v\cdot\nabla v^h = -\nabla^h q\\ \partial_t v^3 - \Delta_\varepsilon v^3 + v\cdot\nabla v^3 = -\varepsilon^2\partial_3 q\\ \operatorname{div} v = 0\\ v_{|t=0} = v_0\end{cases}$$

with $\Delta_\varepsilon \stackrel{\text{def}}{=} \partial_1^2+\partial_2^2+\varepsilon^2\partial_3^2$. As there is no boundary, the rescaled pressure q can be computed with the formula

$$\Delta_\varepsilon q = \sum_{j,k}\partial_j v^k\partial_k v^j = \sum_{j,k}\partial_j\partial_k(v^jv^k). \tag{2.1}$$

It turns out that when ε goes to 0, Δ_ε^{-1} looks like Δ_h^{-1}. In the case of $\mathbb{R}^3$, for low horizontal frequencies, an expression of the type $\Delta_h^{-1}(ab)$ cannot be estimated in L^2 in general. This is the reason why we work in $\mathbb{T}^2\times\mathbb{R}$. In this domain, the problem of low horizontal frequencies reduces to the problem of the horizontal average that we denote by

$$(Mf)(x_3) \stackrel{\text{def}}{=} \overline{f}(x_3) \stackrel{\text{def}}{=} \int_{\mathbb{T}^2} f(x_h,x_3)dx_h.$$

Let us also define $M^\perp f \stackrel{\text{def}}{=} (\operatorname{Id}-M)f$. Notice that, because the vector field v is divergence free, we have $\overline{v}^3\equiv 0$. The system (RNS_ε) can be rewritten in the following form.

$$(RNS_\varepsilon)\quad \begin{cases} \partial_t w^h - \Delta_\varepsilon w^h + M^\perp\big(v\cdot\nabla w^h + w^3\partial_3\overline{v}^h\big) = -\nabla^h q\\ \partial_t w^3 - \Delta_\varepsilon w^3 + M^\perp(v\cdot\nabla w^3) = -\varepsilon^2\partial_3 M^\perp q\\ \partial_t\overline{v}^h - \varepsilon^2\partial_3^2\overline{v}^h = -\partial_3 M(w^3w^h)\\ \operatorname{div}(\overline{v}+w) = 0\\ (\overline{v},w)_{|t=0} = (\overline{v}_0,w_0).\end{cases}$$

The problem to solve this sytem is that there is no obvious way to compensate the loss of one vertical derivative which appears in the equation on w_h and $\overline{v}$ and also, but more hidden, in the pressure term. The method we use is inspired by the one introduced in [2] and can be understood as a global Cauchy-Kowalewski result. This is the reason why the hypothesis of analyticity in the vertical variable is required in our theorem.

Let us denote by $\mathcal{B}$ the unit ball of $\mathbb{R}^3$ and by $\mathcal{C}$ the annulus of small radius 1 and large radius 2. For non negative j, let us denote by L^2_j the space $\mathcal{F}L^2((\mathbb{Z}^2\times\mathbb{R})\cap 2^j\mathcal{C})$ and by L^2_{-1} the space $\mathcal{F}L^2((\mathbb{Z}^2\times\mathbb{R})\cap\mathcal{B})$ respectively equipped with the (semi) norms

$$\|u\|^2_{L^2_j} \stackrel{\text{def}}{=} (2\pi)^{-d}\int_{2^j\mathcal{C}}|\widehat{u}(\xi)|^2d\xi \quad\text{and}\quad \|u\|^2_{L^2_{-1}} \stackrel{\text{def}}{=} (2\pi)^{-d}\int_{\mathcal{B}}|\widehat{u}(\xi)|^2d\xi.$$

Let us now recall the definition of inhomogeneous Besov spaces modeled on L^2.

Definition 2.1. *Let s be a nonnegative real number. The space B^s is the subspace of L^2 such that*

$$\|u\|_{B^s} \stackrel{\text{def}}{=} \big\|\big(2^{js}\|u\|_{L^2_j}\big)_j\big\|_{\ell^1} < \infty.$$

We note that $u \in B^s$ is equivalent to writing $\|u\|_{L^2_j} \leq Cc_j 2^{-js}\|u\|_{B^s}$ where (c_j) is a non negative series which belongs to the sphere of ℓ^1. Let us notice that $B^{\frac{3}{2}}$ is included in $\mathcal{F}(L^1)$ and thus in the space of continuous bounded functions. Moreover, if we substitute ℓ^2 to ℓ^1 in the above definition, we recover the classical Sobolev space H^s.

The theorem we actually prove is the following.

Theorem 2. *Let a be a positive number. There are two positive numbers ε_0 and η such that for any divergence free vector field v_0 satisfying*

$$\|e^{a|D_3|}v_0\|_{B^{\frac{7}{2}}} \leq \eta,$$

then, for any positive ε smaller than ε_0, the initial data

$$u_{0,\varepsilon}(x) \stackrel{\text{def}}{=} \Big(v_0^h(x_h, \varepsilon x_3), \frac{1}{\varepsilon}v_0^3(x_h, \varepsilon x_3)\Big)$$

generates a global smooth solution of (NS) on $\mathbb{T}^2 \times \mathbb{R}$.

2.2. **Definition of the functional setting.**

2.2.1. *Study of a model problem.* In order to motivate the functional setting and to give a flavour of the method used to prove the theorem, let us study for a moment the following simplified model problem for (RNS_ε), in which we shall see in a rather easy way how the same type of method as that of [2] can be used (as a global Cauchy-Kowaleswki technique): the idea is to control a nonlinear quantity, which depends on the solution itself. So let us consider the equation

$$\partial_t u + \gamma u + a(D)(u^2) = 0,$$

where u is a scalar, real-valued function, γ is a positive parameter, and $a(D)$ is a Fourier multiplier of order one. We shall sketch the proof of the fact that if the initial data satisfies, for some positive δ and some small enough constant c,

$$\|u_0\|_X \stackrel{\text{def}}{=} \int e^{\delta|\xi|}|\widehat{u}(\xi)|\,d\xi \leq c\gamma,$$

then one has a global smooth solution, say in the space $\mathcal{F}(L^1)$ as well as all its derivative. The idea of the proof is the following: we want to control the same kind of quantity on the solution, but one expects the radius of analyticity of the solution to decay in time. So let us introduce $\theta(t)$ the "loss of analyticity" of the solution, solving the following ODE:

$$\dot\theta(t) \stackrel{\text{def}}{=} \int e^{(\delta-\lambda\theta(t))|\xi|}|\widehat{u}(\xi)|d\xi \quad \text{with} \quad \theta(0) = 0.$$

The parameter λ will be chosen large enough at the end and we will prove that $\delta - \lambda\theta(t)$ remains positive for all times. The computations that follow hold as long as that assumption is true (and a bootstrap will prove that in fact it does remain true for all times). We define the notation

$$u_\theta(t) = \mathcal{F}^{-1}\left(e^{(\delta-\lambda\theta(t))|\cdot|}\,|\,\widehat{u}(t,\cdot)|\right).$$

Notice that

$$\dot\theta(t) = \|u_\theta(t)\|_{\mathcal{F}(L^1)} \quad \text{and} \quad \theta(t) = \int_0^t \|u_\theta(t')\|_{\mathcal{F}(L^1)}\, dt'. \tag{2.2}$$

Taking the Fourier transform of the equation gives

$$|\widehat{u}(t,\xi)| \leq e^{-\gamma t}|\widehat{u}_0(\xi)| + C\int_0^t e^{-\gamma(t-t')}\,|\xi|\,|\mathcal{F}(u^2)(t',\xi)|\,dt'.$$

Using the fact that

$$(\delta - \lambda\theta(t))\,|\xi| \leq (\delta - \lambda\theta(t'))\,|\xi-\eta| + (\delta - \lambda\theta(t'))\,|\eta| - \lambda|\xi|\int_{t'}^t \dot\theta(t)\,dt''\,,$$

we infer that

$$|\widehat{u}_\theta(t,\xi)| \leq e^{-\gamma t}e^{\delta|\xi|}|\widehat{u}_0(\xi)| + C\int_0^t e^{-\gamma(t-t')-\lambda|\xi|\int_{t'}^t \dot\theta(t)\,dt''}|\xi|\,|\mathcal{F}(u_\theta^2)|(t',\xi)\,dt'.$$

We note the important fact that

$$\int_0^t e^{-\lambda|\xi|\int_{t'}^t \dot\theta(t)\,dt''}|\xi|\,\dot\theta(t')\,dt' \leq \frac{C}{\lambda}\,,$$

which is very useful in what follows. As $\mathcal{F}(ab) = (2\pi)^{-d}(\widehat{a}\star\widehat{b})$, we have, for any $t' \leq t$,

$$|\mathcal{F}(u_\theta^2)|(t',\xi) \leq \Big(\sup_{0\leq t'\leq t}|\widehat{u}_\theta(t',\cdot)|\Big)\star|\widehat{u}_\theta(t',\cdot)|.$$

Recalling that $\mathcal{F}(L^1)$ is an algebra, we infer that

$$\Big\|\sup_{0\leq t'\leq t}|\widehat{u}_\theta(t')|\Big\|_{L^1} \leq \|u_0\|_X + \frac{C}{\lambda}\Big\|\sup_{0\leq t'\leq t}|\widehat{u}_\theta(t')|\Big\|_{L^1}$$

and

$$\theta(t) \leq C_\gamma\Big(\|u_0\|_X + \Big\|\sup_{0\leq t'\leq t}|\widehat{u}_\theta(t')|\Big\|_{L^1}\theta(t)\Big).$$

This allows by bootstrap to obtain the global in time existence of the solution, as soon as the initial data is small enough; we skip the computations, as they will be presented in full detail for the case of the system (RNS_ε).

2.2.2. *Functional setting.* In the light of the computations of the previous section, let us introduce the functional setting we are going to work with to prove the theorem. The proof relies on exponential decay estimates for the Fourier transform of the solution. Thus, for any locally bounded function Ψ on $\mathbb{R}^+\times\mathbb{Z}^2\times\mathbb{R}$ and for any function f, continuous in time and compactly supported in Fourier space, we define

$$(f_\Psi)(t) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(e^{\Psi(t,\cdot)}\widehat{f}(t,\cdot)\big).$$

Now let us define the key quantity we wish to control in order to prove the theorem. In order to do so, let us consider the Friedrichs approximation of the original (NS) system

$$\begin{cases}\partial_t u - \Delta u + \mathbb{P}_n(u\cdot\nabla u + \nabla p) = 0\\ \operatorname{div} u = 0\\ u|_{t=0} = \mathbb{P}_n\, u_{0,\varepsilon},\end{cases}$$

where $\mathbb{P}_n$ denotes the orthogonal projection of L^2 on functions the Fourier transform of which is supported in the ball B_n centered at the origin and of radius n. Thanks to the L^2 energy estimate, this approximated system has a global solution the Fourier transform of which is supported in B_n. Of course, this provides an approximation of the rescaled system namely

$$(RNS_{\varepsilon,n})\quad\begin{cases}\partial_t w^h - \Delta_\varepsilon w^h + \mathbb{P}_{n,\varepsilon}\, M^\perp\big(v\cdot\nabla w^h + w^3\partial_3\overline{v} + \nabla^h q\big) = 0\\ \partial_t w^3 - \Delta_\varepsilon w^3 + \mathbb{P}_{n,\varepsilon}\, M^\perp\big(v\cdot\nabla w^3 + \varepsilon^2\partial_3 q\big) = 0\\ \partial_t\overline{v}^h - \varepsilon^2\partial_3^2\overline{v}^h + \mathbb{P}_{n,\varepsilon}\,\partial_3 M(w^3 w^h) = 0\\ \mathrm{div}(\overline{v}+w) = 0\\ (\overline{v},w)_{|t=0} = (\overline{v}_0, w_0),\end{cases}$$

where $\mathbb{P}_{n,\varepsilon}$ denotes the orthogonal projection of L^2 on functions the Fourier transform of which is supported in $B_{n,\varepsilon} \stackrel{\rm def}{=} \{\xi \,/\, |\xi_\varepsilon|^2 \stackrel{\rm def}{=} |\xi_h|^2 + \varepsilon^2\xi_3^2 \leq n^2\}$. We shall prove analytic type estimates here, meaning exponential decay estimates for the the solution of the above approximated system. In order to make notation not too heavy we will drop the fact that the solutions we deal with are in fact approximate solutions and not solutions of the original system. A priori bounds on the approximate sequence will be derived, which will clearly yield the same bounds on the solution. In the spirit of [2] (see also (2.2) in the previous section), we define the function θ (we drop also the fact that θ depends on ε in all that follows) by

$$\dot\theta(t) = \|w^3_\Phi(t)\|_{B^{\frac72}} + \varepsilon\|w^h_\Phi(t)\|_{B^{\frac72}} \quad\text{and}\quad \theta(0)=0 \tag{2.3}$$

where

$$\Phi(t,\xi) = t^{\frac12}|\xi_h| + a|\xi_3| - \lambda\theta(t)|\xi_3| \tag{2.4}$$

for some λ that will be chosen later on (see Section 2.4). Since the Fourier transform of w is compactly supported, the above differential equation has a unique global solution on $\mathbb{R}^+$. If we prove that

$$\forall t\in\mathbb{R}^+\,,\ \theta(t)\leq\frac{a}{\lambda}\,, \tag{2.5}$$

this will imply that the sequence of approximated solutions of the rescaled system is a bounded sequence of $L^1(\mathbb{R}^+;\mathrm{Lip})$. So is, for a fixed ε, the family of approximation of the original Navier-Stokes equations. This is (more than) enough to imply that a global smooth solution exists.

2.3. **Main steps of the proof.** The proof of Inequality (2.5) will be a consequence of the following two propositions which provide estimates on v^h, w^h and w^3. For technical reasons, these statements require the use of a modified version (introduced in [7]) of $L^\infty_T(B^s)$ spaces.

Definition 2.2. *Let s be a real number. We define the space $\widetilde L^\infty_T(B^s)$ as the subspace of functions f of $L^\infty_T(B^s)$ such that the following quantity is finite:*

$$\|f\|_{\widetilde L^\infty_T(B^s)} \stackrel{\rm def}{=} \sum_j 2^{js}\|f\|_{L^\infty_T(L^2_j)}.$$

Theorem 2 will be an easy consequence of the following propositions, which will be proved in the coming sections.

The first one uses only the fact that the function Φ is subadditive.

Proposition 2.1. *A constant $C_0^{(1)}$ exists such that, for any positive λ, for any initial data v_0, and for any T satisfying $\theta(T) \leq a/\lambda$, we have*

$$\theta(T) \leq \varepsilon \|e^{a|D_3|} w_0^h\|_{B^{\frac{7}{2}}} + \|e^{a|D_3|} w_0^3\|_{B^{\frac{7}{2}}} + C_0^{(1)} \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \theta(T).$$

Moreover, we have the following L^∞-type estimate on the vertical component:

$$\|w_\Phi^3\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|} w_0^3\|_{B^{\frac{7}{2}}} + C_0^{(1)} \|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

The second one is more subtle to prove, and it shows that the use of the analytic-type norm actually allows to recover the missing vertical derivative on v^h, in a L^∞-type space. It should be compared to the methods described in Section 2.2.1.

Proposition 2.2. *A constant $C_0^{(2)}$ exists such that, for any positive λ, for any initial data v_0, and for any T satisfying $\theta(T) \leq a/\lambda$, we have*

$$\|v_\Phi^h\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|} v_0^h\|_{B^{\frac{7}{2}}} + C_0^{(2)} \Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\Big) \|v_\Phi^h\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

2.4. **Proof of the theorem assuming the two propositions.** Let us assume these two propositions are true for the time being and conclude the proof of Theorem 2. It relies on a continuation argument.

For any positive λ and η, let us define

$$\mathcal{T}_\lambda \stackrel{\text{def}}{=} \big\{T \,/\, \max\{\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}, \theta(T)\} \leq 4\eta\big\},$$

As the two functions involved in the definition of $\mathcal{T}_\lambda$ are non decreasing, $\mathcal{T}_\lambda$ is an interval. As θ is an increasing function which vanishes at 0, a positive time T_0 exists such that $\theta(T_0) \leq 4\eta$. Moreover, if $\|e^{a|D|3|} v_0\|_{B^{\frac{7}{2}}} \leq \eta$ then, since $\partial_t v = \mathbb{P}_n F(v)$ (recall that we are considering Friedrich's approximations), a positive time T_1 (possibly depending on n) exists such that $\|v_\Phi\|_{\widetilde{L}^\infty_{T_1}(B^{\frac{7}{2}})} \leq 4\eta$. Thus $\mathcal{T}_\lambda$ is the form $[0, T^\star)$ for some positive $T^\star$. Our purpose is to prove that $T^\star = \infty$. As we want to apply Propositions 2.1 and 2.2, we need that $\lambda\theta(T) \leq a$. This leads to the condition

$$4\lambda\eta \leq a. \tag{2.6}$$

From Proposition 2.1, defining $C_0 \stackrel{\text{def}}{=} C_0^{(1)} + C_0^{(2)}$, we have, for all $T \in \mathcal{T}_\lambda$,

$$\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|} v_0\|_{B^{\frac{7}{2}}} + \frac{C_0}{\lambda} \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} + C_0 \|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

Let us choose $\lambda = \dfrac{1}{2C_0} \cdot$ This gives

$$\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq 2\|e^{a|D_3|} v_0\|_{B^{\frac{7}{2}}} + 4C_0\eta \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

Choosing $\eta = \dfrac{1}{12C_0}$, we infer that, for any $T \in \mathcal{T}_\lambda$,

$$\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq 3\|e^{a|D_3|} v_0\|_{B^{\frac{7}{2}}}. \tag{2.7}$$

Propositions 2.1 and 2.2 imply that, for all $T \in \mathcal{T}_\lambda$,

$$\theta(T) \leq \varepsilon \|e^{a|D_3|} w_0^h\|_{B^{\frac{7}{2}}} + \|e^{a|D_3|} w_0^3\|_{B^{\frac{7}{2}}} + C_0 \eta \theta(T).$$

This implies that

$$\theta(T) \leq 2\varepsilon \|e^{a|D_3|} w_0^h\|_{B^{\frac{7}{2}}} + 2\|e^{a|D_3|} w_0^3\|_{B^{\frac{7}{2}}}.$$

If $2\varepsilon \|e^{a|D_3|} w_0^h\|_{B^{\frac{7}{2}}} + 2\|e^{a|D_3|} w_0^3\|_{B^{\frac{7}{2}}} \leq \eta$ and $\|e^{a|D_3|} v_0^h\|_{B^{\frac{7}{2}}} \leq \eta$, then the above estimate and Inequality (2.7) ensure (2.5). This concludes the proof of Theorem 2.

3. The action of subadditive phases on (para)products

It will be useful to consider, for any function f, the inverse Fourier transform of $|\widehat{f}|$, defined as

$$f^+ \stackrel{\text{def}}{=} \mathcal{F}^{-1} |\widehat{f}|.$$

Let us notice that the map $f \mapsto f^+$ preserves the norm of all B^s spaces. In all this section, Ψ will denote a locally bounded function on $\mathbb{R}^+ \times \mathbb{T}^2 \times \mathbb{R}$ which satisfies the following inequality

$$\Psi(t,\xi) \leq \Psi(t, \xi - \eta) + \Psi(t, \eta). \tag{3.1}$$

In all the following, we will denote by C or c universal constants, which do not depend on any of the parameters of the problem, and which may change from line to line. We will denote generically by c_j any sequence in $\ell^1(\mathbb{Z})$ of norm 1.

We shall denote by $\mathbb{E}_\varepsilon f$ the solution of $\partial_t g - \Delta_\varepsilon g = f$ with initial data equal to 0. We use also a very basic version of Bony's decomposition. Let us define (using the notation introduced in Section 2.1),

$$T_a b \stackrel{\text{def}}{=} \mathcal{F}^{-1} \sum_j \int_{2^j \mathcal{C} \cap \mathcal{B}(\xi, 2^j)} \widehat{a}(\xi - \eta) \widehat{b}(\eta) d\eta \quad \text{and} \quad R_a b \stackrel{\text{def}}{=} \mathcal{F}^{-1} \sum_j \int_{2^j \mathcal{C} \cap \mathcal{B}(\xi, 2^{j+1})} \widehat{a}(\xi - \eta) \widehat{b}(\eta) d\eta.$$

We obviously have $ab = T_a b + R_b a$.

The way the Fourier multiplier e^Ψ acts on bilinear functionals is described by the following lemma.

Lemma 3.1. *For any positive s, a constant C exists which satisfies the following properties. For any function Ψ satisfying (3.1), for any function b in $L^1_T(B^s)$, a positive sequence $(c_j)_{j \in \mathbb{Z}}$ exists in the sphere of $\ell^1(\mathbb{Z})$ such that, for any a in $L^1_T(B^{\frac{3}{2}})$, and any $t \in [0,T]$, we have*

$$\|(T_a b)_\Psi(t)\|_{L^2_j} + \|(R_a b)_\Psi(t)\|_{L^2_j} \leq C c_j 2^{-js} \|a(t)\|_{B^{\frac{3}{2}}} \min\big\{\|b(t)\|_{B^s}, \|b\|_{\widetilde{L}^\infty_T(B^s)}\big\}.$$

Proof. We prove only the lemma for R, the proof for T being strictly identical. Let us first investigate the case when the function Ψ is identically 0. We first observe that for any ξ in the annulus $2^j \mathcal{C}$, we have

$$\mathcal{F}(R_a b(t))(\xi) = \sum_{j' \geq j-2} \int_{2^{j'} \mathcal{C} \cap B(\xi, 2^{j'+1})} \widehat{a}(t, \xi - \eta) \widehat{b}(t, \eta) d\eta.$$

As $B^{\frac{3}{2}}$ is included in $\mathcal{F}(L^1)$, we infer that, by definition of $\|\cdot\|_{\widetilde{L}^\infty_T(B^s)}$,

$$\|R_a b(t)\|_{L^2_j} \leq C\|a(t)\|_{B^{\frac{3}{2}}} \sum_{j'\geq j-2} c_{j'} 2^{-j's} \min\bigl\{\|b(t)\|_{B^s}, \|b\|_{\widetilde{L}^\infty_T(B^s)}\bigr\}.$$

Defining $\widetilde{c}_j = \displaystyle\sum_{j'\geq j-2} 2^{(j-j')s} c_{j'}$ which satisfies $\displaystyle\sum_j \tilde{c}_j \leq C_s$, we obtain

$$\|R_a b(t)\|_{L^2_j} \leq C\tilde{c}_j 2^{-js}\|a(t)\|_{B^{\frac{3}{2}}} \min\bigl\{\|b(t)\|_{B^s}, \|b\|_{\widetilde{L}^\infty_T(B^s)}\bigr\}. \tag{3.2}$$

The lemma is then proved in the case when the function Ψ is identically 0. In order to treat the general case, let us write that

$$\begin{aligned} e^{\Psi(t,\xi)}\mathcal{F}(R_a b)(\xi) &= e^{\Psi(t,\xi)} \sum_j \int_{2^j\mathcal{C}\cap B(\xi,2^j)} \widehat{a}(\xi-\eta)\widehat{b}(\eta)d\eta \\ &\leq \sum_j \int_{2^j\mathcal{C}\cap B(\xi,2^j)} e^{\Psi(t,\xi-\eta)}\widehat{a}^+(\xi-\eta)e^{\Psi(t,\eta)}\widehat{b}^+(\eta)d\eta. \end{aligned}$$

This means exactly that $|\mathcal{F}(R_a b)_\Psi(t,\xi)| \leq \mathcal{F}(R_{a^+_\Psi} b^+_\Psi)(t,\xi)$. Then, the estimate (3.2) implies the lemma. $\square$

Corollary 3.1. *If s is positive, we have, for any function Ψ satisfying (3.1),*

$$\begin{aligned} \|(T_a b)_\Psi\|_{\widetilde{L}^\infty_T(B^s)} + \|(R_a b)_\Psi\|_{\widetilde{L}^\infty_T(B^s)} &\leq C\|a_\Psi\|_{L^\infty_T(B^{\frac{3}{2}})}\|b_\Psi\|_{\widetilde{L}^\infty_T(B^s)} \quad \textit{and} \\ \|(T_a b)_\Psi\|_{L^1_T(B^s)} + \|(R_a b)_\Psi\|_{L^1_T(B^s)} &\leq C\min\bigl\{\|a_\Psi\|_{L^1_T(B^{\frac{3}{2}})}\|b_\Psi\|_{\widetilde{L}^\infty_T(B^s)}, \\ &\qquad\qquad \|a_\Psi\|_{L^\infty_T(B^{\frac{3}{2}})}\|b_\Psi\|_{L^1_T(B^s)}\bigr\}. \end{aligned}$$

Proof. Taking the L^∞ norm in time on the inequality of Lemma 3.1 gives that

$$\|(T_a b)_\Psi\|_{L^\infty_T(L^2_j)} + \|(R_a b)_\Psi\|_{L^\infty_T(L^2_j)} \leq Cc_j 2^{-js}\|a\|_{L^\infty_T(B^{\frac{3}{2}})}\|b\|_{\widetilde{L}^\infty_T(B^s)}.$$

which is the first inequality of the corollary. The proof of the second one is analogous. $\square$

4. The action of the phase Φ on the heat operator

The purpose of this section is the study of the action of the multiplier e^Φ on $\mathbb{E}_\varepsilon f$. Let us recall that the function Φ is defined in (2.4) by $\Phi(t,\xi) = t^{\frac{1}{2}}|\xi_h| + a|\xi_3| - \lambda\theta(t)|\xi_3|$. This action is described by the following lemma.

Lemma 4.1. *A constant C_0 exists such that, for any function f with compact spectrum, we have, for any s,*

$$\begin{aligned} \|(\mathbb{E}_\varepsilon M^\perp f)_\Phi\|_{\widetilde{L}^\infty_T(B^s)} &\leq C_0\|g_\Phi\|_{\widetilde{L}^\infty_T(B^s)} \quad \textit{and} \\ \|(\mathbb{E}_\varepsilon M^\perp f)_\Phi\|_{L^1_T(B^s)} &\leq C_0\|g_\Phi\|_{L^1_T(B^s)} \quad \textit{where} \quad g \overset{\text{def}}{=} \mathcal{F}^{-1}\Bigl(\frac{1}{|\xi_h|}|\mathcal{F}M^\perp f|\Bigr). \end{aligned}$$

Proof. Let us write $\mathbb{E}_\varepsilon$ in terms of the Fourier transform. We have, for any $\xi \in (\mathbb{Z}^2 \setminus \{0\}) \times \mathbb{R}$,

$$\mathcal{F}\left(\mathbb{E}_\varepsilon f\right)_\Phi(t,\xi) = e^{\Phi(t,\xi)} \int_0^t e^{-(t-t')|\xi_\varepsilon|^2} f(t',\xi)dt',$$

with, as in all that follows, $|\xi_\varepsilon|^2 \stackrel{\mathrm{def}}{=} |\xi_h|^2 + \varepsilon^2|\xi_3|^2$. Thus we infer, for any $\xi \in (\mathbb{Z}^2 \setminus \{0\}) \times \mathbb{R}$,

$$|\mathcal{F}\left((\mathbb{E}_\varepsilon f)_\Phi\right)(t,\xi)| \leq \int_0^t e^{-(t-t')|\xi_\varepsilon|^2 + \Phi(t,\xi) - \Phi(t',\xi)} \mathcal{F}(f_\Phi^+)(t',\xi)dt'.$$

By definition of Φ, we have (see [2], estimate (24))

$$\Phi(t,\xi) - \Phi(t',\xi) \leq -\lambda|\xi_3| \int_{t'}^t \dot\theta(t'')dt'' + \frac{t-t'}{2}|\xi_h|^2. \tag{4.1}$$

Thus we have, for any $\xi \in (\mathbb{Z}^2 \setminus \{0\}) \times \mathbb{R}$,

$$|\mathcal{F}\left((\mathbb{E}_\varepsilon f)_\Phi\right)(t,\xi)| \leq \int_0^t e^{-\frac{(t-t')}{2}|\xi_h|^2 - \varepsilon^2(t-t')|\xi_3|^2} \mathcal{F}(f_\Phi^+)(t',\xi)dt'. \tag{4.2}$$

Let us define $\mathcal{C}_h \stackrel{\mathrm{def}}{=} \{1 \leq |\xi_h| \leq 2\} \times \mathbb{R}$. The above inequality means that we have, for any ξ in $2^j\mathcal{C} \cap 2^k\mathcal{C}_h$,

$$|\mathcal{F}((\mathbb{E}_\varepsilon f)_\Phi)(t,\xi)| \leq C \int_0^t e^{-c(t-t')2^{2k}} 2^k \widehat{g}_\Phi(t',\xi)dt'.$$

Taking the L^2 norm in ξ in that inequality gives

$$\|(\mathbb{E}_\varepsilon f)_\Phi(t)\|_{\mathcal{F}L^2(2^j\mathcal{C}\cap 2^k\mathcal{C}_h)} \leq \int_0^t e^{-c(t-t')2^{2k}} 2^k \|g_\Phi(t')\|_{L^2_j} dt'. \tag{4.3}$$

By definition of the $\widetilde{L}^\infty_T(B^s)$ norm, this gives, for any $t \leq T$,

$$\begin{aligned} 2^{js}\|(\mathbb{E}_\varepsilon f)_\Phi\|_{L^\infty_T(L^2(2^j\mathcal{C}\cap 2^k\mathcal{C}_h))} &\leq Cc_j\|g_\Phi\|_{\widetilde{L}^\infty_T(B^s)} \int_0^t e^{-c(t-t')2^{2k}} 2^k dt' \\ &\leq Cc_j 2^{-k}\|g_\Phi\|_{\widetilde{L}^\infty_T(B^s)}. \end{aligned}$$

Now, writing that

$$\|(\mathbb{E}_\varepsilon M^\perp f)_\Phi\|_{L^\infty_T(L^2_j)} \leq \sum_{k=0}^\infty \|\,\mathbb{E}_\varepsilon(f_\Phi)\|_{L^\infty_T(L^2(2^j\mathcal{C}\cap 2^k\mathcal{C}_h))}$$

gives the first inequality of the lemma.

In order to prove the second one, let us use the definition of the norm of the space B^s and (4.3); this gives

$$\begin{aligned} \sum_j 2^{js}\|(\mathbb{E}_\varepsilon f)_\Phi\|_{L^1_T(L^2_j)} &\leq \sum_{j,k} 2^{js}\|\,\mathbb{E}_\varepsilon(f_\Phi)\|_{L^1_T(\mathcal{F}L^2(2^j\mathcal{C}\cap 2^k\mathcal{C}_h))} \\ &\leq C\sum_{j,k}\int_{[0,T]^2} \mathbf{1}_{t\geq t'} e^{-c2^{2k}(t-t')} 2^k c_j(t')\|g_\Phi(t')\|_{B^s} dt'dt. \end{aligned}$$

Integrating first in t gives

$$\sum_j 2^{js}\|(\mathbb{E}_\varepsilon f_\Phi)\|_{L^1_T(L^2_j)} \leq C\sum_{j,k}\int_{[0,T]} 2^{-k} c_j(t')\|g_\Phi(t')\|_{B^s} dt.$$

As the index k is nonnegative, we get the second estimate of the lemma. □

The following lemma is a key one. It is here that the function θ allows the gain of the vertical derivative, in the spirit of the example presented in Section 2.2.1.

Lemma 4.2. *Let $a(D)$ and $b(D)$ be two Fourier multipliers such that $|a(\xi)| \leq C|\xi_3|$ and $|b(\xi)| \leq C|\xi|^2$. We have*

$$\|(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} + \|(\mathbb{E}_\varepsilon\, a(D)T_{b(D)w^3}f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq C\Big(\frac{1}{\lambda} + \|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\Big)\|f_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

Proof. We give only the proof for the first term, the second term is estimated exactly along the same lines. Let us write $\mathbb{E}_\varepsilon$ in Fourier variables. We have

$$\mathcal{F}(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi(t,\xi) = e^{\Phi(t,\xi)}\int_0^t e^{-(t-t')|\xi_\varepsilon|^2}a(\xi)\mathcal{F}(R_{b(D)w^3}f)(t',\xi)dt'.$$

Thus, using that $|a(\xi)| \leq C|\xi_3|$, we obtain

$$|\mathcal{F}(\mathbb{E}_\varepsilon\, a(D)R_{w^3}f)_\Phi(t,\xi)| \leq C\int_0^t e^{-(t-t')|\xi_\varepsilon|^2+\Phi(t,\xi)-\Phi(t',\xi)}|\xi_3|\,|\mathcal{F}((R_{b(D)w^3}f)_\Phi)(t',\xi)|dt'.$$

Taking into account Inequality (4.1), we have

$$|\mathcal{F}(\mathbb{E}_\varepsilon\, a(D)R_{w^3}f)_\Phi(t,\xi)| \leq C\int_0^t e^{-\frac{t-t'}{2}|\xi_\varepsilon|^2-\lambda|\xi_3|\int_{t'}^t\dot{\theta}(t'')dt''}|\xi_3|\,|\mathcal{F}((R_{b(D)w^3}f)_\Phi)(t',\xi)|dt'.$$

Let us denote by Ψ the Fourier multiplier $\Psi a \overset{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{|\xi_h|\leq 2|\xi_3|}\widehat{a})$. If $|\xi_h| \leq 2|\xi_3|$ and ξ is in $2^j\mathcal{C}$, then, we have that $|\xi_3| \sim 2^j$. Thus, we infer that, for any ξ in $2^j\mathcal{C}$,

$$|\mathcal{F}\Psi(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi(t,\xi)| \leq \int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot{\theta}(t'')dt''}2^j|\mathcal{F}((R_{b(D)w^3}f)_\Phi)(t',\xi)|dt'.$$

Taking the L^2 norm gives

$$\|\Psi(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi(t,\cdot)\|_{L^2_j} \leq \int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot{\theta}(t'')dt''}2^j\|(R_{b(D)w^3}f)_\Phi)(t')\|_{L^2_j}dt'.$$

Using Lemma 3.1, we get

$$\begin{aligned}
2^{j\frac{7}{2}}\|\Psi(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi(t,\cdot)\|_{L^2_j} &\leq Cc_j\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \\
&\qquad\times\int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot{\theta}(t'')dt''}2^j\|b(D)w^3_\Phi(t',\cdot)\|_{B^{\frac{3}{2}}}dt' \\
&\leq Cc_j\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \\
&\qquad\times\int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot{\theta}(t'')dt''}2^j\|w^3_\Phi(t',\cdot)\|_{B^{\frac{7}{2}}}dt' \\
&\leq Cc_j\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot{\theta}(t'')dt''}2^j\dot{\theta}(t')dt' \\
&\leq \frac{C}{\lambda}c_j\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.
\end{aligned}$$

By summation in j, we deduce that

$$\|\Psi(\mathbb{E}_\varepsilon\, a(D)R_{w^3}f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq \frac{C}{\lambda}\|f_\Phi(t)\|_{\widetilde{L}^\infty_T(B^{\frac72})}. \tag{4.4}$$

If $2|\xi_3| \leq |\xi_h|$, then, for any ξ in $2^j\mathcal{C}$, $|\xi_h|$ is equivalent to 2^j and $|\xi_3|$ is less than 2^j. So we infer that for any ξ in $2^j\mathcal{C}$,

$$|\mathcal{F}(\mathrm{Id}-\Psi)(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi(t,\xi)| \leq \int_0^t e^{-c(t-t')2^{2j}}2^j|\mathcal{F}((R_{b(D)w^3}f)_\Phi)(t',\xi)|dt'.$$

By definition of $\|\cdot\|_{\widetilde{L}^\infty_T(B^{\frac72})}$, taking the L^2 norm of the above inequality gives

$$\begin{aligned}
2^{j\frac72}\|(\mathrm{Id}-\Psi)(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi(t,\cdot)\|_{L^2_j} &\leq \int_0^t e^{-c2^{2j}(t-t')}2^j2^{j\frac72}\|(R_{b(D)w^3}f)_\Phi)(t')\|_{L^2_j}dt'\\
&\leq Cc_j\|(R_{b(D)w^3}f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.
\end{aligned}$$

After a summation in j, Corollary 3.1 implies that

$$\begin{aligned}
\|(\mathrm{Id}-\Psi)(\mathbb{E}_\varepsilon\, a(D)R_{b(D)w^3}f)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} &\leq C\|b(D)w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac32})}\|f_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\\
&\leq C\|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\|f_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.
\end{aligned}$$

Together with (4.4), this concludes the proof of the lemma. □

Lemma 4.3. *A constant C_0 exists such that, for any function f with compact spectrum, we have for α in $\{1,2\}$,*

$$\begin{aligned}
\big\|\big(\mathbb{E}_\varepsilon(\varepsilon\partial_3)^\alpha M^\perp f\big)_\Phi\big\|_{\widetilde{L}^\infty_T(B^s)} &\leq C_0\|f_\Phi\|_{\widetilde{L}^\infty_T(B^s)} \quad \textit{and}\\
\big\|\big(\mathbb{E}_\varepsilon(\varepsilon\partial_3)^\alpha M^\perp f\big)_\Phi\big\|_{L^1_T(B^s)} &\leq C_0\|f_\Phi\|_{L^1_T(B^s)}.
\end{aligned}$$

Proof. Let us start with the case when $\alpha=2$. Recalling (4.2), we have (for $0<\varepsilon<1$),

$$\varepsilon^2|\mathcal{F}(\mathbb{E}_\varepsilon\,\partial_3^2 f)_\Phi(t,\xi)| \leq \int_0^t e^{-\varepsilon^2\frac{(t-t')}{2}|\xi|^2}\varepsilon^2\xi_3^2\mathcal{F}(f^+_\Phi)(t',\xi)dt'.$$

Writing that $|\xi_3|\leq|\xi|$, we infer that

$$\varepsilon^2\big\|(\mathbb{E}_\varepsilon\,\partial_3^2 f)_\Phi(t)\big\|_{L^2_j} \leq \int_0^t e^{-c\varepsilon^2(t-t')2^{2j}}\varepsilon^2 2^{2j}\|f(t')\|_{L^2_j}dt'.$$

The estimates follow directly by applying Young's inequality in t.

In the case when $\alpha=1$, we decompose f into two parts,

$$f=f^{(1)}+f^{(2)}, \quad \text{with} \quad f^{(1)}=\mathcal{F}^{-1}(\mathbf{1}_{\varepsilon|\xi_3|\leq|\xi_h|}\widehat{f}).$$

Let us start by studying the first contribution. We simply write that

$$\begin{aligned}
\varepsilon\big|\mathcal{F}\big(\mathbb{E}_\varepsilon\,\partial_3 f^{(1)}\big)_\Phi(t,\xi)\big| &\leq \int_0^t e^{-\frac{(t-t')}{2}|\xi_\varepsilon|^2}\varepsilon|\xi_3|\mathcal{F}(f^{(1)}_\Phi)^+(t',\xi)dt'\\
&\leq \int_0^t e^{-\frac{(t-t')}{2}|\xi_\varepsilon|^2}|\xi_h|\mathcal{F}(f^{(1)}_\Phi)^+(t',\xi)dt'
\end{aligned}$$

which amounts exactly to the computation (4.3), with g replaced by $f^{(1)}$. On the other hand, for $f^{(2)}$ we can write

$$\widehat{g}^{(2)}(\xi) \stackrel{\text{def}}{=} \frac{1}{|\xi_h|}\mathbf{1}_{\varepsilon|\xi_3|\geq|\xi_h|}|\mathcal{F}M^\perp f^{(2)}(\xi)|$$

so that

$$\varepsilon\Big|\mathcal{F}\big(\,\mathbb{E}_\varepsilon\,\partial_3 M^\perp f^{(2)}\big)_\Phi(t,\xi)\Big| \leq \int_0^t e^{-\frac{(t-t')}{2}|\xi_h|^2-\varepsilon^2(t-t')|\xi_3|^2}\varepsilon|\xi_3||\xi_h|\widehat{g}^{(2)}(t',\xi)dt'.$$

Since $|\xi_h|\leq\varepsilon|\xi_3|$, we are reduced to the case when $\alpha=2$ and the conclusion comes from the fact that $\|g^{(2)}_\Phi\|_{B^s}\leq\|M^\perp f^{(2)}_\Phi\|_{B^s}\leq\|f_\Phi\|_{B^s}$. That proves the lemma. □

5. Classical analytic-type parabolic estimates

The purpose of this section is to prove Proposition 2.1. We shall use the algebraic structure of the Navier-Stokes system and the fact that the function Φ is subadditive.

Let us first bound the horizontal component. We recall that

$$w^h_\Phi(t) = e^{t\Delta_\varepsilon+\Phi(t,D)}w^h(0) - \big(\mathbb{E}_\varepsilon\,M^\perp(v\cdot\nabla w^h)\big)_\Phi(t) - \big(\mathbb{E}_\varepsilon\,M^\perp(w^3\partial_3\overline{v}^h)\big)_\Phi(t) - \big(\mathbb{E}_\varepsilon(\nabla_h q)\big)_\Phi(t).$$

We note that $v\cdot\nabla w^h = \mathrm{div}_h(v^h\otimes w^h)+\partial_3(w^3w^h)$, recalling that $v^3=w^3$. On the one hand, using Lemma 4.1 and Corollary 3.1, we can write

$$\begin{aligned}\varepsilon\|\,\mathbb{E}_\varepsilon(\mathrm{div}_h(v^hw^h))_\Phi\|_{L^1_T(B^{\frac72})} &\leq C\varepsilon\|(v^hw^h)_\Phi\|_{L^1_T(B^{\frac72})}\\ &\leq C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\varepsilon\|w^h\|_{L^1_T(B^{\frac72})}.\end{aligned}$$

By definition of θ, we infer that

$$\varepsilon\|\,\mathbb{E}_\varepsilon(\mathrm{div}_h(v^hw^h))_\Phi\|_{L^1_T(B^{\frac72})}\leq C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}. \tag{5.1}$$

On the other hand, Lemma 4.3 and Corollary 3.1 imply that

$$\begin{aligned}\|\,\mathbb{E}_\varepsilon\left(\varepsilon\partial_3M^\perp(w^3w^h)\right)_\Phi\|_{L^1_T(B^{\frac72})} &\leq C\|w^3w^h\|_{L^1_T(B^{\frac72})}\\ &\leq C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\end{aligned} \tag{5.2}$$

For the second term, we use paradifferential calculus which gives

$$\begin{aligned}w^3\partial_3\overline{v}^h &= T_{w^3}\partial_3\overline{v}^h + R_{\partial_3\overline{v}^h}w^3\\ &= \partial_3T_{w^3}\overline{v}^h - T_{\partial_3w^3}\overline{v}^h + R_{\partial_3\overline{v}^h}w^3.\end{aligned}$$

Using again Lemma 4.3 and Corollary 3.1, we get

$$\begin{aligned}\|\,\mathbb{E}_\varepsilon\left(\varepsilon\partial_3M^\perp T_{w^3}\overline{v}^h\right)_\Phi\|_{L^1_T(B^{\frac72})} &\leq C\|(T_{w^3}\overline{v}^h)_\Phi\|_{L^1_T(B^{\frac72})}\\ &\leq C\|w^3_\Phi\|_{L^1_T(B^{\frac72})}\|\overline{v}^h\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\end{aligned}$$

By definition of θ, we infer

$$\|\,\mathbb{E}_\varepsilon\left(\varepsilon\partial_3M^\perp T_{w^3}\overline{v}^h\right)_\Phi\|_{L^1_T(B^{\frac72})}\leq C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}. \tag{5.3}$$

By Lemma 4.1 and Corollary 3.1, we can write that

$$\begin{aligned}\|\,\mathbb{E}_\varepsilon\,\big(\varepsilon M^\perp T_{\partial_3 w^3}\overline{v}^h)_\Phi\|_{L^1_T(B^{\frac72})} &\leq C\varepsilon\|(T_{\partial_3 w^3}\overline{v}^h)_\Phi\|_{L^1_T(B^{\frac72})}\\ &\leq C\varepsilon\|w^3_\Phi\|_{L^1_T(B^{\frac72})}\|\overline{v}^h\|_{\widetilde{L}^\infty_T(B^{\frac72})}\end{aligned}$$

so that

$$\|\,\mathbb{E}_\varepsilon\,\big(\varepsilon M^\perp T_{\partial_3 w^3}\overline{v}^h)_\Phi\|_{L^1_T(B^{\frac72})} \leq C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}, \tag{5.4}$$

and finally along the same lines we have

$$\|\,\mathbb{E}_\varepsilon\,\big(\varepsilon M^\perp R_{\partial_3\overline{v}^h} w^3)_\Phi\|_{L^1_T(B^{\frac72})} \leq C\theta(T)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}. \tag{5.5}$$

Now we are left with the study of the pressure. Some of its properties are described in the following lemma.

Lemma 5.1. *Let us define $\nabla_\varepsilon \stackrel{\text{def}}{=} (\nabla_h, \varepsilon\partial_3)$. The following two inequalities on the rescaled pressure hold:*

$$\begin{aligned}\varepsilon\|(\mathbb{E}_\varepsilon\,\nabla_\varepsilon M^\perp q)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} &\leq C\|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac72})} \quad \text{and}\\ \varepsilon\|(\mathbb{E}_\varepsilon\,\nabla_\varepsilon M^\perp q)_\Phi\|_{L^1_T(B^{\frac72})} &\leq C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\theta(T).\end{aligned}$$

Proof. Using the formula (2.1) on the rescaled pressure and the divergence free condition on v, let us decompose it as $\varepsilon q = q_{1,\varepsilon} - q_{2,\varepsilon}$ with

$$\begin{aligned}q_{1,\varepsilon} &\stackrel{\text{def}}{=} \sum_{k=1}^2 \partial_k\partial_\ell\Delta_\varepsilon^{-1}(\varepsilon w^k v^\ell) + \sum_{1\leq k\leq 2}\partial_k(\varepsilon\partial_3)\Delta_\varepsilon^{-1}(w^3v^k) \quad \text{and}\\ q_{2,\varepsilon} &\stackrel{\text{def}}{=} 2\varepsilon\partial_3\Delta_\varepsilon^{-1}(w^3\,\mathrm{div}_h\,w^h).\end{aligned}$$

Let us start with $q_{1,\varepsilon}$. We have

$$\nabla_\varepsilon q_{1,\varepsilon} = \sum_{k=1}^2\partial_k\bigg(\sum_{\ell=1}^2\nabla_\varepsilon\partial_\ell\Delta_\varepsilon^{-1}(\varepsilon w^k v^\ell) + \nabla_\varepsilon(\varepsilon\partial_3)\Delta_\varepsilon^{-1}(w^3v^k)\bigg).$$

As $\nabla_\varepsilon^2\Delta_\varepsilon^{-1}$ is a family of bounded Fourier multipliers (uniformly with respect to ε), we infer from Lemma 4.1 and Corollary 3.1 that

$$\varepsilon\|(\mathbb{E}_\varepsilon(\nabla_\varepsilon M^\perp q_{1,\varepsilon}))_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C\|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac72})} \quad \text{and} \tag{5.6}$$

$$\varepsilon\|(\mathbb{E}_\varepsilon(\nabla_\varepsilon M^\perp q_{1,\varepsilon}))_\Phi\|_{L^1_T(B^{\frac72})} \leq C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\theta(T). \tag{5.7}$$

In order to study $q_{2,\varepsilon}$, let us observe that

$$\begin{aligned}w^3\,\mathrm{div}_h\,w^h &= R_{\mathrm{div}_h\,w^h}w^3 + T_{w^3}\,\mathrm{div}_h\,w^h\\ &= R_{\mathrm{div}_h\,w^h}w^3 + \sum_{k=1}^2\big(\partial_k T_{w^3}w^k - T_{\partial_k w^3}w^k\big).\end{aligned} \tag{5.8}$$

As above we get, using Lemma 4.1 and Corollary 3.1,

$$\begin{aligned}\varepsilon\|(\mathbb{E}_\varepsilon(\nabla_\varepsilon M^\perp q_{2,\varepsilon}))_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} &\leq C\|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \quad \text{and}\\ \varepsilon\|(\mathbb{E}_\varepsilon(\nabla_\varepsilon M^\perp q_{2,\varepsilon}))_\Phi\|_{L^1_T(B^{\frac{7}{2}})} &\leq C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\theta(T).\end{aligned}$$

Together with estimates (5.6) and (5.7), this concludes the proof of the lemma. □

The above Lemma 5.1, together with estimates (5.1) to (5.4), implies that

$$\varepsilon\|w^h\|_{L^1_T(B^{\frac{7}{2}})} \leq \varepsilon\|e^{a|D_3|}w^h_0\|_{B^{\frac{7}{2}}} + C_0^{(1)}\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\theta(T). \tag{5.9}$$

Let us prove the estimates on the vertical component. It turns out that it is better behaved because of the special structure of the system. Indeed, thanks to the divergence free condition, almost no vertical derivatives appear in the equation of w^3: we have (since $w^3 = v^3$)

$$\partial_t w^3 - \Delta_\varepsilon w^3 = -v^h\cdot\nabla_h w^3 + w^3 \operatorname{div}_h w^h - \varepsilon^2\partial_3 q. \tag{5.10}$$

The Duhamel formula reads

$$w^3(t) = e^{t\Delta_\varepsilon}w^3(0) + \mathbb{E}_\varepsilon\, M^\perp(w^3\operatorname{div}_h w^h - v^h\cdot\nabla_h w^3)(t) - \mathbb{E}_\varepsilon\, M^\perp(\varepsilon^2\partial_3 q)(t).$$

Applying the Fourier multiplier $e^{\Phi(t,D)}$ to the above relation gives

$$w^3_\Phi(t) = e^{t\Delta_\varepsilon+\Phi(t,D)}w^3(0) + \big(\mathbb{E}_\varepsilon\, M^\perp(w^3\operatorname{div}_h w^h - v^h\cdot\nabla_h w^3)\big)_\Phi(t) - \big(\mathbb{E}_\varepsilon\, M^\perp\varepsilon^2\partial_3 q\big)_\Phi(t). \tag{5.11}$$

Using (5.8) and then Lemma 4.1 and Corollary 3.1, we get

$$\big\|\big(\mathbb{E}_\varepsilon\, M^\perp(w^3\operatorname{div}_h w^h)\big)_\Phi\big\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq C\|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \quad \text{and} \tag{5.12}$$

$$\big\|\big(\mathbb{E}_\varepsilon\, M^\perp(w^3\operatorname{div}_h w^h)\big)_\Phi\big\|_{L^1_T(B^{\frac{7}{2}})} \leq C\|w^3_\Phi\|_{L^1_T(B^{\frac{7}{2}})}\|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}. \tag{5.13}$$

Writing that

$$\begin{aligned} v^h\cdot\nabla_h a &= \sum_{k=1}^2\big(T_{v^k}\partial_k a + R_{\partial_k a}v^k\big)\\ &= -T_{\operatorname{div}_h w^h}a + \sum_{k=1}^2\big(\partial_k T_{v^k}a + R_{\partial_k a}v^k\big)\end{aligned}$$

and using Lemma 4.1 and Corollary 3.1, we get

$$\begin{aligned}\big\|\big(\mathbb{E}_\varepsilon\, M^\perp(v^h\cdot\nabla_h w^3)\big)_\Phi\big\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} &\leq C\|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \quad \text{and}\\ \big\|\big(\mathbb{E}_\varepsilon\, M^\perp(v^h\cdot\nabla_h w^3)\big)_\Phi\big\|_{L^1_T(B^{\frac{7}{2}})} &\leq C\|w^3_\Phi\|_{L^1_T(B^{\frac{7}{2}})}\|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.\end{aligned}$$

Together with estimates (5.12) and (5.13), and Lemma 5.1, this gives

$$\begin{aligned}\|w^3\|_{L^1_T(B^{\frac{7}{2}})} &\leq \|e^{a|D_3|}w^3_0\|_{B^{\frac{7}{2}}} + C_0^{(1)}\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\theta(T) \quad \text{and}\\ \|w^3_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} &\leq \|e^{a|D_3|}w^3_0\|_{B^{\frac{7}{2}}} + C_0^{(1)}\|v_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.\end{aligned}$$

Together with (5.9), this concludes the proof of Proposition 2.1.

6. The gain of one vertical derivative on the horizontal part

In this section we shall prove Proposition 2.2. The proof will be separated into two parts: first we shall consider the case of the horizontal average $\overline{v}_\Phi^h$, and then the remainder w_Φ^h.

6.1. The gain of one vertical derivative on the horizontal average.

We shall study in this section the equation on the horizontal average of the solution. We emphasize that in the equation on $\overline{v}$ we cannot recover the vertical derivative appearing in the force term by the regularizing effect. The fundamental idea to gain a vertical derivative is to use the analyticity of the solution and therefore to estimate $\overline{v}_\Phi$. The lemma is the following.

Lemma 6.1. *A constant C_0 exists such that, for any positive λ, for any initial data v_0, and for any T satisfying $\theta(T) \leq a/\lambda$, we have*

$$\|\overline{v}_\Phi^h\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq \|e^{a|D_3|}\overline{v}_0^h\|_{B^{\frac{7}{2}}} + C_0\Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\Big)\|v_\Phi^h\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}.$$

Proof. The horizontal average $\overline{v}$ satisfies

$$\partial_t\overline{v} - \varepsilon^2\partial_3^2\overline{v} = -\partial_3 M(w^3w^h) \quad \text{and} \quad \overline{v}_{|t=0} = \overline{v}_0. \tag{6.1}$$

Let us define $G \stackrel{\text{def}}{=} -\partial_3 M(w^3 w^k)$. Writing the solution of (6.1) in terms of the Fourier transform, we get, using (4.1) with $\xi_h = 0$,

$$|\mathcal{F}(\overline{v}_\Phi)(t,\xi)| \leq |\mathcal{F}\overline{v}_0(\xi)|e^{a|\xi_3|} + \int_0^t e^{-\lambda|\xi_3|\int_{t'}^t \dot\theta(t'')dt''}|\mathcal{F}(G_\Phi)(t',\xi)|dt'.$$

Then, taking the L^2_j norm, we infer that

$$\|\overline{v}_\Phi(t)\|_{L^2_j} \leq \|e^{a|D_3|}\overline{v}_0\|_{L^2_j} + \int_0^t e^{-c\lambda 2^j\int_{t'}^t \dot\theta(t'')dt''}\|G_\Phi(t')\|_{L^2_j}dt'. \tag{6.2}$$

Now, let us estimate $\|G_\Phi(t')\|_{L^2_j}$. For any function a, using the fact that the vector field w is divergence free, let us write that

$$\begin{aligned}
\partial_3(w^3a) &= \partial_3\big(T_{w^3}a + R_a w^3\big)\\
&= \partial_3 T_{w^3}a + R_{\partial_3 a}w^3 - R_a \operatorname{div}_h w^h\\
&= \partial_3 T_{w^3}a + R_{\partial_3 a}w^3 - \sum_{\ell=1}^2 \partial_\ell R_a w^\ell + \sum_{\ell=1}^2 R_{\partial_\ell a}w^\ell.
\end{aligned} \tag{6.3}$$

Thus, we infer that

$$\begin{aligned}
G &= -\partial_3 M T_{w^3}w^k - M\Big(R_{\partial_3 w^k}w^3 + \sum_{\ell=1}^2 R_{\partial_\ell w^k}w^\ell - \sum_{\ell=1}^2 \partial_\ell R_{w^3}w^\ell\Big)\\
&= -\partial_3 M T_{w^3}w^k - M\Big(R_{\partial_3 w^k}w^3 + \sum_{\ell=1}^2 R_{\partial_\ell w^k}w^\ell\Big).
\end{aligned} \tag{6.4}$$

Now, let us study $\mathcal{F}M(T_a b)_\Phi$ and $\mathcal{F}M(R_a b)_\Phi$ for two functions a and b which have 0 horizontal average. As the two terms are identical, let us study the first one. By definition, we have

$$\mathcal{F}\big(T_a b)(t,(0,\xi_3)) = \sum_j \int_{2^j\mathcal{C}\cap B((0,\xi_3),2^j)} \widehat{a}((0,\xi_3)-\eta)\widehat{b}(\eta)d\eta.$$

As $\theta(T) \leq \lambda^{-1}a$, by definition of Φ we have, for any $\eta \in (\mathbb{Z}^2\setminus\{0\})\times\mathbb{R}$,

$$\begin{aligned} \Phi(t,(0,\xi_3)) &\leq \Phi(t,(0,\xi_3-\eta_3)) + \Phi(t,(0,\eta_3)) \\ &\leq -2t^{\frac12} + \Phi(t,((0,\xi_3)-\eta) + \Phi(t,-\eta). \end{aligned}$$

Thus we have

$$|(\mathcal{F}M(T_a b)_\Phi)(t,\xi)| \leq e^{-2t^{\frac12}}(\mathcal{F}MT_{a^+_\Phi} b^+_\Phi)(t,\xi).$$

Applied to (6.4), this implies that

$$\begin{aligned} \big|\mathcal{F}G_\Phi(t,\xi)\big| &\leq |\xi_3|\mathcal{F}\big(T_{(w^3_\Phi)^+}(w^k_\Phi)^+\big)(t,(0,\xi_3)) \\ &\qquad + e^{-2t^{\frac12}}\mathcal{F}\Big(R_{(\partial_3 w^k_\Phi)^+}(w^3_\Phi)^+ + \sum_{\ell=1}^2 R_{(\partial_\ell w^k_\Phi)^+}(w^\ell_\Phi)^+\Big)(t,(0,\xi_3)). \end{aligned}$$

Inequality (3.1) then implies that, for any $t\in[0,T]$,

$$2^{j\frac72}\|G_\Phi(t)\|_{L^2_j} \leq Cc_j\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\big(2^j\|w^3_\Phi(t)\|_{B^{\frac72}} + e^{-2t^{\frac12}}\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\big). \tag{6.5}$$

Then, by definition of θ, Inequalities (6.2) and (6.5) imply that

$$\begin{aligned} 2^{j\frac72}\|(\overline{v}_\Phi)(t)\|_{L^2_j} &\leq 2^{j\frac72}\|e^{a|D_3|}\overline{v}_0\|_{L^2_j} \\ &\quad + Cc_j\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\bigg(\int_0^t e^{-c\lambda 2^j\int_{t'}^t\dot\theta(t'')dt''}2^j\dot\theta(t')dt' + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\int_0^t e^{-2t'^{\frac12}}dt'\bigg). \end{aligned}$$

This gives

$$2^{j\frac72}\|\overline{v}_\Phi\|_{L^\infty_T(L^2_j)} \leq 2^{j\frac72}\|e^{a|D_3|}\overline{v}_0\|_{L^2_j} + Cc_j\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\Big).$$

Taking the sum over j concludes the proof of the lemma. □

6.2. **The gain of the vertical derivative on the whole horizontal term.** Now let us estimate the rest of the horizontal term, that is $\|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}$. As in Section 6.1, the function θ will play a crucial role.

Lemma 6.2. *A constant C_0 exists such that, for any λ, for any initial data v_0, and for any T satisfying $\theta(T)\leq a/\lambda$, we have*

$$\|w^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq \|e^{a|D_3|}w^h_0\|_{B^{\frac72}} + C_0\Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\Big)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.$$

Proof. The Duhamel formula writes

$$w^h(t) = e^{t\Delta_\varepsilon}w^h(0) - \mathbb{E}_\varepsilon\,\mathrm{div}_h(v^h\otimes v^h)(t) - \mathbb{E}_\varepsilon\,M^\perp\partial_3(w^3v^h)(t) - \mathbb{E}_\varepsilon(\nabla_h q)(t).$$

Lemma 4.1 and Corollary 3.1 imply that

$$\begin{aligned}\|(\mathbb{E}_\varepsilon \operatorname{div}_h(v^h\otimes v^h))_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} &\leq C\|(v^h\otimes v^h)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\\ &\leq C\|v^h_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac72})}.\end{aligned}\tag{6.6}$$

Then using (6.3) we get, thanks to Leibnitz formula,

$$\begin{aligned}M^\perp\partial_3(w^3v^k) &= M^\perp\partial_3 T_{w_3}v^k + F^k \quad\text{with}\\ F^k &\stackrel{\text{def}}{=} M^\perp\Big(R_{\partial_3 v^k}w^3 - \sum_{\ell=1}^2\big(\partial_\ell R_{v^k}w^\ell - R_{\partial_\ell v_k}w^\ell\big)\Big).\end{aligned}$$

Thanks to Lemma 4.1 and Corollary 3.1, we get

$$\begin{aligned}\|(\mathbb{E}_\varepsilon\, M^\perp F^k)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} &\leq \|(F^k)_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\\ &\leq C\|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\end{aligned}$$

Together with Lemma 4.2, this gives

$$\big\|\big(\mathbb{E}_\varepsilon\, M^\perp\partial_3(w^3v^h)\big)_\Phi\big\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C_0\Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}\Big)\|v^h_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})}.\tag{6.7}$$

Now let us study the pressure term. Formula (2.1) together with the divergence free condition leads to the decomposition $q = q_h(w^h) + q_3(v)$ with

$$q_h(w^h) \stackrel{\text{def}}{=} \Delta_\varepsilon^{-1}\Big((\operatorname{div}_h w^h)^2 + \sum_{1\leq k,\ell\leq 2}\partial_k w^\ell\partial_\ell w^k\Big) \quad\text{and}\tag{6.8}$$

$$q_3(v) \stackrel{\text{def}}{=} \Delta_\varepsilon^{-1}\Big(\sum_{1\leq \ell\leq 2}\partial_3 v^\ell\partial_\ell w^3\Big).\tag{6.9}$$

For the first term we use Bony's decomposition in order to obtain

$$\partial_k w^\ell\partial_\ell w^k = T_{\partial_k w^\ell}\partial_\ell w^k + R_{\partial_\ell w^k}\partial_k w^\ell.$$

Then the Leibnitz formula implies that

$$\partial_k w^\ell\partial_\ell w^k = \partial_\ell T_{\partial_k w^\ell}w^k + \partial_k R_{\partial_\ell w^k}w^\ell - T_{\partial_\ell\partial_k w^\ell}w^k - R_{\partial_k\partial_\ell w^k}w^\ell.\tag{6.10}$$

On the other hand, again by paradifferential calculus, we can write that

$$\begin{aligned}(\operatorname{div}_h w^h)^2 &= T_{\operatorname{div}_h w^h}\operatorname{div}_h w^h + R_{\operatorname{div}_h w^h}\operatorname{div}_h w^h\\ &= \operatorname{div}_h\big(T_{\operatorname{div}_h w^h}w^h + R_{\operatorname{div}_h w^h}w^h\big) - \sum_{k=1}^2\Big(T_{\partial_k\operatorname{div}_h w^h}w^k + R_{\partial_k\operatorname{div}_h w^h}w^k\Big).\end{aligned}\tag{6.11}$$

Then Lemma 4.1 implies that

$$\|(\mathbb{E}_\varepsilon\,\nabla_h q_h(w^h))_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C_0\|(M^\perp q_h(w^h))_\Phi\|_{\widetilde{L}^\infty(B^{\frac72})}.$$

Using Corollary 3.1 and the fact that the operators $\nabla_h\Delta_\varepsilon^{-1}M^\perp$ and $\Delta_\varepsilon^{-1}M^\perp$ are bounded (uniformly in ε) Fourier multipliers, we obtain

$$\big\|\big(\mathbb{E}_\varepsilon(\nabla_h q_h(w^h))\big)_\Phi\big\|_{\widetilde{L}^\infty_T(B^{\frac72})} \leq C_0\|v^h_\Phi\|^2_{\widetilde{L}^\infty_T(B^{\frac72})}.\tag{6.12}$$

For the second term, let us decompose $q_3(v)$ in the following way:

$$\begin{aligned}\partial_3 v^\ell \partial_\ell w^3 &= T_{\partial_3 v^\ell}\partial_\ell w^3 + R_{\partial_\ell w^3}\partial_3 v^\ell\\ &= \partial_\ell T_{\partial_3 v^\ell} w^3 + \partial_3 R_{\partial_\ell w^3} v^\ell - T_{\partial_3\partial_\ell v^\ell} w^3 - R_{\partial_3\partial_\ell w^3} v^\ell.\end{aligned}$$

Using now Lemma 4.1 together with Corollary 3.1 and Lemma 4.2, we obtain

$$\big\|\big(\mathbb{E}_\varepsilon(\nabla_h q_3(v))\big)_\Phi\big\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})} \leq C_0\Big(\frac{1}{\lambda} + \|v_\Phi\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}\Big)\|v^h\|_{\widetilde{L}^\infty_T(B^{\frac{7}{2}})}. \tag{6.13}$$

The expected result is obtained putting together estimates (6.12) and (6.13) on the pressure with estimates (6.6) and (6.7) on the nonlinear terms. □

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